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A LIMIT THEOREM FOR THE PERRON–FROBENIUS OPERATOR OF TRANSFORMATIONS ON $[0,1]$ WITH INDIFFERENT FIXED POINTS

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ABSTRACT

A limit theorem is proved for $\left\{\sum_{k=0}^{n-1} P^k\right\}_{n=1}^{\infty}$, where P is the Perron–Frobenius operator associated with transformations on the unit interval with indifferent fixed points.

1. Introduction

In [9] a limit theorem is obtained for $\{\sum_{k=0}^{n-1} P^k\}_{n=1}^{\infty}$, where P is the Perron–Frobenius operator associated with transformations T on the unit interval with an indifferent fixed point at $x = 0$. The local behaviour of T at 0 is assumed to be of the form

$$T(x) = x + ax^2 + o(x^2) \quad \text{with } a > 0.$$

As a consequence the absolutely continuous invariant measure is infinite.

The purpose of the present paper is to prove a theorem of this type for transformations T on $[0,1]$ with finitely many indifferent fixed points under more general conditions on the local behaviour of T at these points. They are merely assumed to be regular sources giving rise to an infinite invariant measure. The class of transformations treated here is the same as in [21].

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In section 2 we introduce the necessary definitions and notations and recall some known facts. Section 3 contains the basic estimates and the main result. In section 4 we use J. Aaronson's method to determine the normalizing sequences for some examples.

For transformations as considered in [9] with

$$T(x) = x + ax^{d+1} + o(x^{d+1}) \quad (x \to 0),$$

where $0 < d \le 1$, results on the rate of convergence are obtained in [11]. The resulting invariant measure is finite for $d < 1$.

2. Preliminaries

Let $\xi_1 = \{B(k)\colon k \in I\}$ be a collection of pairwise disjoint subintervals of $[0,1]$ such that $\lambda\left(\bigcup_{k\in I} B(k)\right) = 1$, where λ denotes the Lebesgue measure on the σ-field $\mathcal{B}$ of Lebesgue measurable subsets of [0,1]. We consider transformations on [0,1] satisfying the following conditions (cf. [21]).

(1) $T|_{B(k)}$ is twice differentiable, and $\overline{TB(k)} = [0,1]$ for all $k \in I$.
There is a non-empty finite set $J \subseteq I$ such that each $B(j)$, $j \in J$, contains a fixed point x_j with $T'(x_j) = 1$.

(2) $|T'| \ge \rho(\varepsilon) > 1$ on $\bigcup_{k\in I} B(k) \setminus \bigcup_{j\in J}(x_j - \varepsilon, x_j + \varepsilon)$ for each $\varepsilon > 0$.

(3) For $j \in J$, T' is decreasing on $(x_j - \eta, x_j) \cap B(j)$ and increasing on $(x_j, x_j + \eta) \cap B(j)$ for some $\eta > 0$.

(4) $T''/(T')^2$ is bounded on $\bigcup_{k\in I} B(k)$.

In particular, $T|_{B(k)}$ has a C^1-extension to $\overline{B(k)}$ for every $k \in I$, and condition (2) is equivalent to

(2)' $T' > 1$ on $\overline{B(j)} \setminus \{x_j\}$ for $j \in J$, and $|T'| \ge \rho$ on $\bigcup_{k\in I\setminus J} B(k)$
with $\rho > 1$.

We use the notations

$$B(k_1, \ldots, k_n) = \bigcap_{i=1}^{n} T^{-i+1}(B(k_i)), \quad (k_1, \ldots, k_n) \in I^n,$$

$$\xi_n = \{B(k_1, \ldots, k_n)\colon (k_1, \ldots, k_n) \in I^n\}, \quad n \ge 1.$$

For $Z = B(k_1, \ldots, k_n)$, $f_Z \equiv f_{k_1,\ldots,k_n}$ denotes the C^1-extension of $(T^n|_Z)^{-1}$ to $[0,1]$.

According to the results in [21], T is conservative and exact with respect to λ and admits an invariant measure m equivalent to λ such that the density $dm/d\lambda$ has a version of the form

$$h(x) = h_0(x) \prod_{j \in J} \frac{x - x_j}{x - f_j(x)}, \quad x \in [0,1] \setminus \{x_j : j \in J\},$$

where h_0 is continuous and positive on $[0,1]$. Since f_j'' is bounded on $(0,1)$ this formula implies that m is infinite.

Let $P: L_1(\lambda) \to L_1(\lambda)$ denote the Perron–Frobenius operator for T with respect to λ, defined by the relation

$$\int_A Pu\, d\lambda = \int_{T^{-1}(A)} u\, d\lambda \quad \text{for all } u \in L_1(\lambda) \text{ and all } A \in \mathcal{B}.$$

In our case $P^n (n \geq 1)$ is given by

$$P^n u = \sum_{Z \in \xi_n} u \circ f_Z \cdot |\, f_Z' \,|\,.$$

Since T is exact and m is infinite,

$$(\star) \qquad \lim_{n \to \infty} \int_A P^n u\, d\lambda = 0$$

holds for all $u \in L_1(\lambda)$ and all $A \in \mathcal{B}$ with $m(A) < \infty$.

To see this, let $u \in L_1(\lambda)$ be non-negative and let B be a measurable set with $0 < m(B) < \infty$. Putting

$$v = \frac{\int u\, d\lambda}{m(B)}\, h \cdot 1_B$$

we have

$$\int_A P^n u\, d\lambda \leq \| P^n u - P^n v \|_1 + \frac{\int u\, d\lambda}{m(B)} m(B \cap T^{-n} A).$$

Since T is exact and $\int (u - v) d\lambda = 0$,

$$\lim_{n \to \infty} \| P^n u - P^n v \|_1 = 0.$$

Due to the invariance of m

$$m(B \cap T^{-n} A) \leq m(A),$$

and therefore

$$\overline{\lim} \int_A P^n u \, d\lambda \le \frac{\int u \, d\lambda}{m(B)} m(A).$$

Since $m(B)$ may be chosen arbitrarily large, $(\star)$ follows.

As an immediate consequence, $P^n u \to 0$ in measure with respect to λ for each u in $L_1(\lambda)$. Thus $P^n u$ tends to become small in this sense with increasing n. In order to obtain non-trivial limit theorems this tendency has to be compensated by suitable normalizations.

3. The main result

THEOREM: *Let T satisfy the conditions (1) - (4). Then there exists a sequence $\{a_n\}$ of positive numbers such that for all Riemann-integrable functions u on $[0,1]$*

$$\frac{1}{a_n} \sum_{k=0}^{n-1} P^k u \to \left(\int u \, d\lambda \right) h$$

uniformly on compact subsets of $[0,1] \setminus \{x_j \colon j \in J\}$, where h is a version of the invariant density of T continuous on $[0,1] \setminus \{x_j \colon j \in J\}$.

The basic estimates are contained in the following Lemmas. Throughout T is assumed to satisfy the conditions (1) - (4).

We introduce the notations

$$G_j(x) = \frac{x - x_j}{x - f_j(x)}, \quad x \in [0,1] \setminus \{x_j\}, \quad j \in J,$$

$$\alpha = \min\{\alpha_1, 1 - \alpha_2\}, \quad \text{where}$$

$$\alpha_1 = \min\{x_j \colon j \in J,\ x_j > 0\}, \quad \alpha_2 = \max\{x_j \colon j \in J,\ x_j < 1\},$$

and, for $x \in (0, \alpha)$,

$$G(x) = \max\{G^-(x), G^+(x)\}, \quad \text{where}$$

$$G^-(x) = \max\{G_j(x_j - x) \colon j \in J,\ x_j > 0\},$$

$$G^+(x) = \max\{G_j(x_j + x) \colon j \in J,\ x_j < 1\},$$

with obvious versions if $\{x_j \colon j \in J\} = \{0\}$ or $\{1\}$.

Furthermore, let

$$A_x = [0,1] \setminus \bigcup_{j \in J} (x_j - x, x_j + x) \quad \text{for } x > 0.$$

We follow the convention to put

$$f_{k_s,\dots,k_n} = id, \quad \text{and consequently} \quad f'_{k_s,\dots,k_n} = 1,$$

if $s > n$.

LEMMA 1: *There exists a constant K_0 such that for all $x \in (0,\alpha)$, for all $n \geq 1$ and all $(k_1,\dots,k_n) \in I^n$*

$$\sum_{s=1}^{n+1} |\, f'_{k_s,\dots,k_n}(t)\,| \leq K_0 G(x) \quad \textit{for } t \in A_x\,.$$

Proof: Choose $\eta > 0$, ρ as in (2)', $\vartheta \in \left[\frac{1}{\rho},1\right)$ and $\delta \in (0,\eta)$ such that the following conditions hold for each $j \in J$:

$$\begin{aligned}
&(x_j-\eta, x_j+\eta) \cap [0,1] \subseteq B(j,j),\\
&f'_j \quad \text{is increasing on} \quad (x_j-\eta, x_j) \cap [0,1] \quad \text{and}\\
&\qquad \text{decreasing on} \;\; (x_j, x_j+\eta) \cap [0,1],\\
&f'_j \leq \vartheta \quad \text{on} \quad [0,1] \setminus (x_j-\eta, x_j+\eta), \quad \text{and}\\
&f'_j(x_j-\delta) \geq \vartheta \quad \text{if} \quad x_j > 0, \quad \text{and}\\
&f'_j(x_j+\delta) \geq \vartheta \quad \text{if} \quad x_j < 1.
\end{aligned}$$

We note first that

$$(\star) \qquad \begin{aligned} |\, f'_k(t)| &\leq \vartheta, \quad \text{if } k \notin J \text{ and } t \in [0,1], \quad \text{and}\\ f'_j(t) &\leq \vartheta, \quad \text{if } j \in J \text{ and } t \notin B(j). \end{aligned}$$

In the first case, by condition (2)'

$$|\, f'_k(t)| = \frac{1}{|\, T'(f_k(t))|} \leq \frac{1}{\rho} \leq \vartheta;$$

in the second case, $t \in [0,1] \setminus (x_j-\eta, x_j+\eta)$.

Furthermore, for all $x \in (0,\delta)$, all $j \in J$ and all $n \geq 1$,

$$(\star\star) \qquad (f_j^n)'(t) \leq \begin{cases} (f_j^n)'(x_j-x), & t \in [0, x_j-x],\\ (f_j^n)'(x_j+x), & t \in [x_j+x, 1]. \end{cases}$$

We verify the first estimate. If $t \in [0, x_j-\eta]$,

$$f'_j(t) \leq \vartheta \leq f'_j(x_j-\delta) \leq f'_j(x_j-x),$$

where the last step follows from the convexity of f_j on $(x_j - \eta, x_j)$. By the same reason

$$f_j'(t) \le f_j'(x_j - x) \quad \text{for } t \in (x_j - \eta, x_j - x].$$

This proves the assertion for $n = 1$. For the general case we use the chain rule

$$(f_j^n)'(t) = \prod_{s=0}^{n-1} f_j'(f_j^s(t)).$$

If $t \in [0, x_j - x]$, $f_j^s(t) \in [0, f_j^s(x_j - x)]$ and $f_j^s(x_j - x) = x_j - x'$ with $x' \in (0, \delta)$. Therefore we can apply the case $n = 1$ to obtain

$$f_j'(f_j^s(t)) \le f_j'(f_j^s(x_j - x)),$$

and thus

$$(f_j^n)'(t) \le (f_j^n)'(x_j - x).$$

Now we fix $x \in (0, \delta)$, $t \in A_x$, $n \ge 1$, and $(k_1, \dots, k_n) \in I^n$. Let $m \in \mathbb{N}_0$ denote the number of indices $s \in \{1, \dots, n\}$ for which

$$k_s \notin J$$

or

$$k_s \in J \quad \text{and} \quad k_{s+1} \ne k_s.$$

If $m \ge 1$, let $1 \le i_1 < \cdots < i_m \le n$ denote these indices, and put $i_0 = 0$, $i_{m+1} = n + 1$ for all $m \ge 0$. Then, including the case $m = 0$,

$$\sum_{s=1}^{n+1} |f'_{k_s,\dots,k_n}(t)| = \sum_{r=1}^{m+1} \sum_{s=i_{r-1}+1}^{i_r} |f'_{k_s,\dots,k_n}(t)|.$$

For $i_{r-1} < s \le i_r$,

$$f_{k_s,\dots,k_n} = f_j^{i_r - s} \circ f_{k_{i_r},\dots,k_n} \quad \text{for some } j \in J,$$

hence

$$f'_{k_s,\dots,k_n}(t) = \left(f_j^{i_r - s}\right)' \left(f_{k_{i_r},\dots,k_n}(t)\right) f'_{k_{i_r},\dots,k_n}(t).$$

Taking into account that $|f_k'| \le 1$ for all $k \in I$ we obtain using $(\star)$

$$\begin{aligned} |f'_{k_{i_r},\dots,k_n}(t)| &= \prod_{i=i_r}^{n} \left| f'_{k_i}\left(f_{k_{i+1},\dots,k_n}(t)\right) \right| \\ &\le \prod_{\nu=r}^{m} \left| f'_{k_{i_\nu}}\left(f_{k_{i_\nu+1},\dots,k_n}(t)\right) \right| \\ &\le \vartheta^{m-r+1}. \end{aligned}$$

Again by the definition of the indices i_ν, $f_{k_{i_r},\dots,k_n}(t) \notin B(j,j)$ if $r \le m$, and $f_{k_{i_r},\dots,k_n}(t) = t$ if $r = m+1$. In both cases $f_{k_{i_r},\dots,k_n}(t) \in [0,1] \setminus (x_j - x, x_j + x)$. Assume $f_{k_{i_r},\dots,k_n}(t) \in [0, x_j - x]$. Then by (⋆⋆)

$$\left(f_j^{i_r-s}\right)'\left(f_{k_{i_r},\dots,k_n}(t)\right) \le \left(f_j^{i_r-s}\right)'(x_j - x),$$

and we obtain using the Lemma in [20], p. 305

$$\begin{aligned}\sum_{s=i_{r-1}+1}^{i_r} \mid f'_{k_s,\dots,k_n}(t)\mid &\le \vartheta^{m-r+1} \sum_{s=i_{r-1}+1}^{i_r} \left(f_j^{i_r-s}\right)'(x_j - x)\\ &\le \vartheta^{m-r+1} G_j(x_j - x)\\ &\le \vartheta^{m-r+1} G(x).\end{aligned}$$

The same bound results, if $f_{k_{i_r},\dots,k_n}(t) \in [x_j + x, 1]$. Hence we have

$$\sum_{s=1}^{n+1} \mid f'_{k_s,\dots,k_n}(t)\mid \le \sum_{r=1}^{m+1} \vartheta^{m-r+1} G(x) \le \frac{G(x)}{1-\vartheta}.$$

Choosing a suitable constant K_0 we obtain the estimate for all $x \in (0, \alpha)$. ∎

The main step in the proof of the theorem is to show the asserted convergence for functions u of the form

$$u = \sum_{Z \in \mathcal{A}} \mid f'_Z \mid,$$

where $\mathcal{A}$ is a non-empty subclass of ξ_n for some fixed n. The arguments in the proof of the following Lemma show that these functions are continuous on $[0,1]$ and have bounded derivative on $(0,1)$. Moreover, $u > 0$ on $[0,1]$. This is the reason why we deal first with functions of this type (cf. also [15]).

We introduce the following notation:

$$u_n = P^n u, \quad n \ge 0.$$

LEMMA 2: *Let u be continuous and positive on $[0,1]$ and differentiable on $(0,1)$, and let u' be bounded on $(0,1)$. Then u_n $(n \ge 0)$ is of the same type, and there exists a constant $K = K(u)$ such that*

$$\mid u'_n \mid \le K\, G(x) \cdot u_n \quad \text{on } A_x \cap (0,1)$$

for all $n \ge 0$ and all $x \in (0, \alpha)$.

Proof: Formal differentiation of

$$u_n = \sum_{Z\in\xi_n} u\circ f_Z \cdot f_Z' \cdot \sigma_Z, \qquad \sigma_Z = \mathrm{sign} f_Z',$$

yields

$$(\star) \qquad u_n' = \sum_{Z\in\xi_n} u'\circ f_Z \cdot (f_Z')^2 \cdot \sigma_Z + \sum_{Z\in\xi_n} u\circ f_Z \cdot f_Z'' \cdot \sigma_Z.$$

For $Z = B(k_1,\ldots,k_n)$,

$$(\star\star) \qquad \begin{aligned} \left|\frac{f_Z''}{f_Z'}\right| &= \left|\sum_{j=1}^{n} \frac{f_{k_j}''\circ f_{k_{j+1},\ldots,k_n}}{f_{k_j}'\circ f_{k_{j+1},\ldots,k_n}} f_{k_{j+1},\ldots,k_n}'\right| \\ &\le M\cdot \sum_{j=1}^{n} \mid f_{k_{j+1},\ldots,k_n}' \mid \quad \text{on } (0,1), \end{aligned}$$

where M is a bound of $\mid T''\mid /(T')^2$ according to condition (4).

Since $\mid f_{k_{j+1},\ldots,k_n}' \mid \le 1$ we have for some constant $\beta = \beta_n$

$$\mid f_Z' \mid, \mid f_Z'' \mid \le \beta \cdot \lambda(Z) \quad \text{on } (0,1) \quad \text{for all } Z\in\xi_n.$$

Therefore u_n is of the same type as u, and u_n' is given by $(\star)$. Thus,

$$|u_n'| \le c\,u_n + \sum_{Z\in\xi_n} u\circ f_Z \cdot \mid f_Z''| \quad \text{on } (0,1),$$

where $c = \sup_{t\in(0,1)} (\mid u'(t)\mid /u(t))$.

Now let $x\in(0,\alpha)$ and $t\in A_x\cap(0,1)$. Using Lemma 1 we obtain from $(\star\star)$

$$\mid f_Z''(t)\mid \le MK_0 G(x) \mid f_Z'(t)\mid,$$

and therefore

$$\begin{aligned} \mid u_n'(t)\mid &\le (c + MK_0\,G(x))\,u_n(t) \\ &\le (c+MK_0)\,G(x) u_n(t). \qquad \blacksquare \end{aligned}$$

By the usual mean value argument we have as an immediate consequence:

For each function u as in Lemma 2, for all $x\in(0,\alpha)$, all $n\ge 0$ and all t_1, t_2 in the same component E of A_x,

$$u_n(t_1) \le e^{c(x)}\, u_n(t_2) \quad \text{with } c(x) = K\,G(x).$$

These estimates give a deeper insight into the asymptotic behaviour of the sequence $\{u_n\}$. For, by integration, we obtain for all $t \in E$

$$e^{-c(x)} \int_E u_n \, d\lambda \;\leq\; \lambda(E) \cdot u_n(t) \;\leq\; e^{c(x)} \int_E u_n \, d\lambda, \quad n \geq 0.$$

The upper estimate and the remarks in section 2 show that $\{u_n\}$ and $\{u_n'\}$ tend to 0 uniformly on $A_x \cap (0,1)$. Since T is conservative and ergodic, the lower estimate shows that $\left\{\sum_{k=0}^{n-1} u_k\right\}$ tends to infinity uniformly on A_x.

Proof of the Theorem: Let $E_1, \ldots, E_s$ denote the components of $[0,1] \setminus \{x_j \colon j \in J\}$. Choose $t_i \in E_i$, $1 \leq i \leq s$, such that $T(t_i) = t_1$, $2 \leq i \leq s$, and $\varepsilon \in (0, \alpha)$ such that $t_i \in A_\varepsilon \cap E_i$, $1 \leq i \leq s$. Let further

$$\kappa = \max \{ | T'(t_i) | : 2 \leq i \leq s \} .$$

Suppose first u is as in Lemma 2. According to the above estimates we have for every $x \in (0, \varepsilon)$ and every $k \geq 0$

$$u_k \leq e^{c(x)} \, u_k(t_i) \text{ and } |u_k'| \leq c(x) e^{c(x)} \, u_k(t_i) \quad \text{on } A_x \cap (0,1) \cap E_i \, , \quad 1 \leq i \leq s.$$

For $2 \leq i \leq s$ we have in addition

$$u_{k+1}(t_1) = (P u_k)(t_1) = \sum_{t \colon T(t) = t_1} \frac{u_k(t)}{|T'(t)|} \;\geq\; \kappa^{-1} u_k(t_i).$$

Therefore the functions

$$U_n = \left(\sum_{k=0}^{n-1} u_k \right) \Big/ \left(\sum_{k=0}^{n-1} u_k(t_1) \right) \quad (n \geq 1)$$

are uniformly bounded on A_x and Lipschitz on each component of A_x with a common constant $L(x)$.

Let h denote the version of the invariant density which is continuous on $[0,1] \setminus \{x_j \colon j \in J\}$ and normalized by $h(t_1) = 1$. A diagonalization argument based on the Theorem of Arzelà–Ascoli shows that each subsequence of $\{U_n\}$ has a subsequence converging to a continuous function uniformly on compact subsets of $[0,1] \setminus \{x_j \colon j \in J\}$. On the other hand, if a subsequence of $\{U_n\}$ converges to a function g uniformly on compact subsets of $[0,1] \setminus \{x_j \colon j \in J\}$, then $Pg = g$ and hence $g = h$ by the uniqueness of h. Therefore the sequence $\{U_n\}$ tends to h pointwise on $[0,1] \setminus \{x_j \colon j \in J\}$. Finally, a 3ε-argument shows that this convergence is uniform on A_x for each $x \in (0, \varepsilon)$.

By the Chacon–Ornstein Theorem for every two functions u, v of this type

$$\sum_{k=0}^{n-1} u_k(t_1) \sim \frac{\int u\,d\lambda}{\int v d\lambda} \sum_{k=0}^{n-1} v_k(t_1) \quad (n \to \infty).$$

Hence we can choose one fixed sequence $\{a_n\}$ of positive numbers to obtain

$$\frac{1}{a_n} \sum_{k=0}^{n-1} u_k \to \left(\int u\,d\lambda \right) \cdot h$$

uniformly on A_x for all $x \in (0, \varepsilon)$ and all functions u considered so far.

The rest of the proof is an approximation procedure.

First note that the asserted limiting behaviour also holds for functions v on $[0,1]$ such that $v = u$ on the open interval $(0,1)$ and u is a above.

As already stated, if $\mathcal{A}$ is a non-empty class of cylinders of some fixed order, the function

$$u = \sum_{Z \in \mathcal{A}} | f'_Z |$$

satisfies the conditions of Lemma 2.

The last two remarks imply that

$$\frac{1}{a_n} \sum_{k=0}^{n-1} P^k 1_A \to \lambda(A) \cdot h$$

uniformly on A_x for all $x \in (0, \varepsilon)$, if A is an interval whose endpoints are endpoints of cylinders of a given order or accumulation points thereof.

Finally, if u is a non-negative Riemann-integrable function, we approximate u from below and from above by step functions which are constant on intervals of the preceding type. This then finishes the proof of the Theorem. ∎

4. Examples

In order to obtain the sequence $\{a_n\}$ in case the behaviour of T at the fixed points $\{x_j\colon j \in J\}$ is sufficiently specified, we follow J. Aaronson [3]. We may assume that $\{a_n\}$ is increasing.

Let

$$V(t) = \sum_{n=0}^{\infty} (a_{n+1} - a_n) t^n, \quad t \in (0,1) \quad (a_0 = 0).$$

According to the terminology in [3] our theorem implies that any set $A \in \mathcal{B}$ of positive measure which is bounded away from x_j for each $j \in J$ and satisfies $\lambda(\partial A) = 0$ is a Darling–Kac set, and

$$(m(A))^2 \cdot a_n \sim \sum_{k=0}^{n-1} m\left(A \cap T^{-k}(A)\right) \quad (n \to \infty).$$

Thus for any such set A the asymptotic renewal equation in [3] gives

$$V(t) \sim \frac{1}{(1-t)Q(t)} \quad (t \to 1),$$

where

$$Q(t) = \sum_{n=0}^{\infty} q_n t^n$$

with

$$\sum_{k=0}^{n-1} q_k = m\left(\bigcup_{k=0}^{n-1} T^{-k}(A)\right) =: L_A(n), \quad n \geq 1.$$

It is shown in [21], Theorem 3, that the order of $\{L_A(n)\}$ does not depend on A.

Example 1: Suppose first that $\{x_j : j \in J\} \subseteq \{0, 1\}$, and

$$T(x) = x \pm a_j(x - x_j)^2 + o\left((x - x_j)^2\right) \quad (x \to x_j)$$

with $a_j > 0$, $j \in J$. Then the invariant density has the form

$$h(x) = g(x) \prod_{j \in J} \mid x - x_j \mid^{-1}$$

with g continuous and positive on $[0, 1]$, and

$$\sum_{k=0}^{n-1} q_k \sim c \cdot \log n \quad (n \to \infty)$$

with $c = \sum_{j \in J} g(x_j)$ ([21], Theorem 4). Hence,

$$Q(t) \sim c \cdot \log \frac{1}{1-t} \quad (t \to 1),$$

and therefore

$$V(t) \sim \frac{1}{c(1-t)\log \frac{1}{1-t}} \quad (t \to 1).$$

The application of a Tauberian Theorem (see e.g. [10]) yields

$$a_n \sim \frac{1}{c} \cdot \frac{n}{\log n} \qquad (n \to \infty)$$

(cf. [9], [11]).

Example 2: Let T satisfy (1) – (4) and assume that for each $j \in J$

$$T(x) = x \pm a_j | x - x_j |^{p_j+1} + o\left(| x - x_j |^{p_j+1} \right) \quad (x \to x_j),$$

where $a_j > 0$, $p_j \geq 1$, and $\max\{p_j : j \in J\} > 1$.

The invariant density is given by

$$h(x) = g(x) \prod_{j \in J} | x - x_j |^{-p_j} \quad (g \text{ continuous and positive on } [0,1]).$$

With the notations

$$\begin{aligned} p &= \max\{p_j : j \in J\}, \quad \alpha = \frac{1}{p}, \\ J_0 &= \{j \in J : p_j = p\}, \\ \varepsilon(x) &= 2 - 1_{\{0,1\}}(x), \quad \text{and} \\ c_j &= g(x_j) \prod_{i \in J, i \neq j} | x_j - x_i |^{-p_i}, \quad j \in J, \end{aligned}$$

we have

$$\sum_{k=0}^{n-1} q_k \sim \frac{c}{1-\alpha} \cdot n^{1-\alpha} \quad (n \to \infty),$$

where

$$c = \frac{1}{p^\alpha} \sum_{j \in J_0} \varepsilon(x_j)\, c_j\, a_j^{1-\alpha}.$$

By the same procedure as in Example 1 we obtain

$$a_n \sim \frac{1}{c} \cdot \frac{\sin \pi\alpha}{\pi\alpha} \cdot n^\alpha \quad (n \to \infty).$$

Example 3: Let $f(0) = 0$, $f(x) = x + x^2 \cdot e^{-1/x}$, $x > 0$, and let $a \in (0,1)$ be determined by $f(a) = 1$. Define T on $[0,1]$ by

$$T(x) = \begin{cases} f(x), & x \in [0,a], \\ \frac{x-a}{1-a}, & x \in (a,1] \quad (\text{cf. [21], p. 94}). \end{cases}$$

For this map

$$h(x) = g(x)e^{\frac{1}{x}}/x \quad (g \text{ continuous and positive on } [0,1]),$$

and

$$\sum_{k=0}^{n-1} q_k \sim g(0) \cdot \frac{n}{\log n} \quad (n \to \infty).$$

Hence

$$a_n \sim \frac{1}{g(0)} \cdot \log n \quad (n \to \infty).$$

Example 4 and correction to [20]*:* For $r > 0$ let the map $T\colon \mathbb{R} \to \mathbb{R}$ be given by

$$\begin{aligned} T(x) &= x - \frac{1}{x^r}, & x > 0, \\ T(x) &= -T(-x), & x < 0. \end{aligned}$$

In [16] it is shown that for each $r > 0$ the transformation T is conservative and exact with respect to λ and admits an invariant measure $m \sim \lambda$ such that the density h_T has the form

$$h_T(x) = h_0(x)\,(1+\mid x \mid)^{r-1/r}, \quad x \in \mathbb{R},$$

where h_0 is continuous and bounded away from 0 and ∞ on $\mathbb{R}$. In particular, m is infinite if and only if

$$r \geq \frac{\sqrt{5}-1}{2}.$$

The case $r = 1$ is the well-known Boole's transformation with $h_T \equiv 1$.

The cases $r = 2n+1$, $n \in \mathbb{N}_0$, are also considered in [20]. The order of the invariant density stated there is not correct for $n \geq 1$. The error comes from the fact that for $n \geq 1$ the conjugated map on $[0,1]$ considered in [20] has slope 0 at the point $\frac{1}{2}$ and hence does not satisfy the crucial condition (T1) in [20]. According to the above representation of h_T the correct order for $r = 2n+1$ is

$$r - \frac{1}{r} = 2n\frac{2n+2}{2n+1}.$$

In the following let r be in the interval $[\frac{\sqrt{5}-1}{2}, \infty)$. To obtain a suitable conjugate on $[0,1]$ we use the function

$$\varphi(x) = \int_{-\infty}^{x} \psi(t)\,dt, \quad x \in \mathbb{R},$$

where

$$\psi(t) = c_r \cdot (3 + |\, t\,|^{r+1})^{-1/r}, \quad t \in \mathbb{R},$$

and c_r is chosen in such a way that $\varphi(\infty) = 1$. Let then $S : [0,1] \to [0,1]$ be given by $S = \varphi T \varphi^{-1}$.

We claim that S satisfies our conditions (1) – (4) with

$$S(x) = x + ax^{p+1} + o\left(x^{p+1}\right) \quad (x \to 0),$$

where $p = r(r+1)$ and $a = 1/r(rc_r)^p$. To see this we analyse the branch $S|_{[\frac{1}{2},1]}$.

For $x \in \left(\frac{1}{2}, 1\right)$,

$$S'(x) = G(\varphi^{-1}(x)),$$

where

$$G(y) = \frac{(\psi \circ T)(y) \cdot T'(y)}{\psi(y)} = G_1(y^{r+1}),$$

and

$$G_1(z) = (r+z)\left(\frac{3+z}{3z^r + |\, z-1|^{\,r+1}}\right)^{1/r} \quad (y, z > 0).$$

These formulae show that S has a C^1-extension to $[\frac{1}{2}, 1]$ with $S'(1) = 1$ and a C^2-extension to $[\frac{1}{2}, 1)$.

In order to see that $S' > 1$ on $[\frac{1}{2}, 1)$ we distinguish three cases. If $r \geq 1$, for all $z \geq 0$

$$(r+z)^r(3+z) \geq (1+z^r)(3+z) > 3z^r + z^{r+1} + 1 \geq 3z^r + |\, z-1|^{\,r+1},$$

i.e. $G_1(z) > 1$. If $z \geq 1$, for all $r > 0$

$$(r+z)^r(3+z) > z^r \cdot (3+z) > 3z^r + |\, z-1|^{\,r+1},$$

hence $G_1(z) > 1$. If $r < 1$ and $z \in [0,1]$ we argue as follows. The function $f(z) = 3\left((r+z)^r - z^r\right)$ is decreasing on $[0,1]$, and

$$f(1) = 3\left((r+1)^r - 1\right) > 1 \quad \text{since } r \geq \frac{\sqrt{5}-1}{2}.$$

Hence

$$3\left((r+z)^r - z^r\right) > 1 \geq (1-z)^{r+1} - z(r+z)^r, \quad z \in [0,1],$$

or, equivalently, $G_1(z) > 1$.

From the relation

$$\frac{d}{dz}\left((G_1(z)\right)^r = \frac{(r+z)^{r-1}}{(3z^r + |\,z-1|^{\,r+1})^2}\left\{3z^{r+1} + 4r|\,z-1|^{\,r+1} - \right.$$
$$\left. -(r+1)(r+4)z|\,z-1|^{\,r}\operatorname{sign}(z-1) + o(z^{r+1})\right\} \quad (z \to \infty)$$

we obtain

$$\lim_{z\to\infty} z^2 G_1'(z) = -\frac{r^2+r+1}{r}.$$

Since $G_1'(z) < 0$ for z sufficiently large, S' is decreasing in a neighbourhood of $x = 1$.

Finally, putting $p = r(r+1)$ we obtain

$$\begin{aligned}
(p+1)\lim_{x\to 1}\frac{S(x)-x}{(1-x)^{p+1}} &= \lim_{x\to 1}\frac{1-S'(x)}{(1-x)^p} \\
&= \lim_{y\to\infty}\frac{1-G(y)}{(1-\varphi(y))^p} \\
&= \lim_{y\to\infty}\left(\frac{y^{-1/r}}{1-\varphi(y)}\right)^p \cdot \lim_{y\to\infty}\frac{G'(y)}{(r+1)y^{-r-2}} \\
&= \left(\frac{1}{rc_r}\right)^p \lim_{z\to\infty} z^2 \cdot G_1'(z) \\
&= -\frac{p+1}{r(rc_r)^p}.
\end{aligned}$$

In particular, S has a C^2-extension to $\left[\frac{1}{2}, 1\right]$, and hence condition (4) also holds. This concludes the proof of the asserted properties of S.

Now there exists a sequence $\{a_n\}$ of positive numbers such that for all Riemann-integrable functions u on $[0,1]$

$$\frac{1}{a_n}\sum_{k=0}^{n-1} P_S^k\, u \to \left(\int u\, d\lambda\right) h_S$$

uniformly on compact subsets of $(0,1)$, where P_S is the Perron–Frobenius operator for S and h_S is a version of the invariant density of S of the form

$$h_S(x) = \frac{g(x)}{x^p(1-x)^p} \quad \text{with } g \text{ continuous and positive on } [0,1].$$

According to the previous examples,

$$a_n \sim \begin{cases} \frac{1}{2g(0)}\cdot\frac{n}{\log n}, & r = \frac{\sqrt{5}-1}{2} \\[2ex] \frac{1}{2c}\cdot\frac{\sin \pi\alpha}{\pi\alpha}\cdot n^{\alpha}, & r > \frac{\sqrt{5}-1}{2}, \end{cases}$$

where $\alpha = 1/p$ and $c = g(0)a^{1-\alpha}/p^{\alpha}$.

Carrying over these results to the map T we obtain for the Perron–Frobenius operator P_T of T:

$$\frac{1}{a_n}\sum_{k=0}^{n-1} P_T^k v \to \left(\int_{\mathbb{R}} v\, d\lambda\right) h_T$$

uniformly on compact subsets of $\mathbb{R}$ for all functions v on $\mathbb{R}$ which are Riemann-integrable on each compact interval and satisfy

$$v(x) = O\left(\mid x \mid^{-\frac{r+1}{r}}\right) \quad \text{as } \mid x \mid \to \infty.$$

Here $h_T = (h_S \circ \varphi) \cdot \varphi'$. In terms of the transformation T the constant c is given by $c = h_0(\infty)/(r+1)^{\alpha}$, where $h_0(\infty) = \lim_{x\to\infty} h_0(x)$.

For the special case $r = 1$ compare with results in [1].

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